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Criteria Convolutions When Combining the Solutions of the Multicriteria Axial Assignment Problem

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Abstract—This paper is devoted to a classical NP-hard problem, known as the three-index axial assignment problem. Within the corresponding framework, the problem of combining feasible solutions is posed as an assignment problem on the set of solutions containing only the components of selected feasible solutions. The issues of combining solutions for the multicriteria problem with different criteria convolutions are studied. In the general case, the combination problem turns out to be NP-hard. Polynomial solvability conditions are obtained for the combination problem.

Keywords: axial assignment problem, multi-index problems, combining solutions, polynomial solvability, NP-hardness

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1. INTRODUCTION

There is a wide class of applied problems formalized by multi-index axial assignment problems [1–4]. In the general case (no additional constraints), the class of multi-index axial assignment problems is NP-hard [5]. Particular polynomially solvable subclasses and subclasses with polynomial approximation algorithms are known [1, 6, 7]. Axial assignment problems with a special structure of the multi-index cost matrix were studied in [6–10]; the branch-and-bound method for solving the axial assignment problem was described in [11]; a parallel implementation of the solution algorithm was discussed in [12]; a genetic algorithm was developed in [13]; a lower bound for the problem was considered in [14]; asymptotically optimal solutions were investigated in [15]. Multicriteria formulations of multi-index assignment problems were addressed in [16–19]. The problem of combining solutions of the axial assignment problem was investigated in [20–22].

This paper is devoted to the multicriteria three-index axial assignment problem. Within the corresponding framework, we formulate the problem of combining feasible solutions as a multicriteria assignment problem on the set of solutions obtained by combining the components of given feasible solutions. The convolution of criteria is considered as a compromise scheme for solving multicriteria problems. We consider linear, minimax, and lexicographic convolutions.

For these types of convolutions, the problems of combining N feasible solutions are studied. The following results are established below:

—In the linear convolution case, the combination problem is polynomially solvable for $N = 2$ and NP-hard for $N \geqslant 4$.

—In the lexicographic convolution case, the combination problem is polynomially solvable for $N = 2$ and NP-hard for $N \geqslant 4$.

—In the minimax convolution case, the combination problem is NP-hard for $N \geqslant 2$.

This research continues the series of papers [20–22], where combination was studied when solving the (single-criterion) axial assignment problem. Here, combining solutions means constructing a solution that contains only assignments from selected feasible solutions. In turn, solving the combination problem involves searching for an optimal solution, in terms of a selected criterion or compromise scheme, on the set of solutions obtained by combining the selected feasible solutions.

Note that the original multicriteria three-index axial assignment problem with the considered types of convolutions as a compromise scheme is NP-hard. Thus, the combination algorithms proposed below for the polynomially solvable cases can be applied as a complement to heuristic or approximate algorithms for solving the original NP-hard problems. Many heuristic or approximate algorithms for solving assignment problems are based on constructing a series of feasible solutions with further record selection among them; for example, see [1, 3, 6]. As an alternative to the conventional approach of selecting a record among the feasible solutions found, we use a step of combining the constructed feasible solutions. In this case, the solution obtained by combination is surely not worse than the record in terms of the optimization criterion: the resulting set of combination-based solutions also contains all feasible solutions found initially.

The remainder of the paper is organized as follows. Section 2 presents the formal statement of the multicriteria three-index axial assignment problem and describes the types of criteria convolutions used. In Section 3, the problem of combining feasible solutions is posed. In Sections 4, 5, and 6, we investigate combination problems for the linear, lexicographic, and minimax criteria convolutions, respectively. Section 7 provides the results of computational experiments.

2. THE MULTICRITERIA THREE-INDEX AXIAL ASSIGNMENT PROBLEM

Let I, J, and K be disjoint index sets, $I \cap J = \emptyset$, $I \cap K = \emptyset$, $J \cap K = \emptyset$, and $|I| = |J| = |K| = n$; M is a fixed number of problem criteria; c^u_{ijk}, where $i \in I$, $j \in J$, $k \in K$, and $u = \overline{1, M}$, are three-index cost matrices; x_{ijk}, where $i \in I$, $j \in J$, and $k \in K$, is the three-index matrix of the variables. Then the multicriteria three-index axial assignment problem is formulated as the following integer linear programming problem:

$$\sum_{i\in I}\sum_{j\in J} x_{ijk} = 1, \quad k \in K, \tag{1}$$

$$\sum_{i\in I}\sum_{k\in K} x_{ijk} = 1, \quad j \in J, \tag{2}$$

$$\sum_{j\in J}\sum_{k\in K} x_{ijk} = 1, \quad i \in I, \tag{3}$$

$$x_{ijk} \in \{0, 1\}, \quad i \in I, \quad j \in J, \quad k \in K, \tag{4}$$

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c^u_{ijk} x_{ijk} \to \min, \quad u = \overline{1, M}. \tag{5}$$

For the sake of convenience, let Z_M denote the multicriteria problem (1)–(5). As is known, in the single-criterion formulation (for $M = 1$), the assignment problem Z_M is NP-hard [5].

In the multicriteria formulation, we will consider the convolution of criteria as a compromise scheme. The linear, lexicographic, and minimax convolutions will be studied below.

Given weights α_u, $u = \overline{1, M}$, the linear convolution of criteria has the form

$$\sum_{u=1}^{M} \alpha_u \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c^u_{ijk} x_{ijk} \to \min\,. \tag{6}$$

For the sake of convenience, let Z_L denote problem (1)–(4), (6). Obviously, the NP-hard (single-criterion) three-index axial assignment problem is polynomially reducible to the problem Z_L. Indeed, consider the three-index axial assignment problem with a cost matrix c_{ijk}, where $i \in I$, $j \in J$, and $k \in K$. In the corresponding problem Z_L, we define $c^1_{ijk} = c_{ijk}$, $c^u_{ijk} = 0$, $\alpha_1 = 1$, $\alpha_u = 0$, where $i \in I$, $j \in J$, $k \in K$, and $u = \overline{2, M}$. Then the following result is true.

Proposition 1. *The problem Z_L is NP-hard.*

Assume that the order of significance of the criteria (5) coincides with their original indexing. We introduce a preference relation on the set of feasible solutions of the assignment problem. Let P denote the set of feasible solutions of the system of constraints (1)–(4). For $x^1, x^2 \in P$, we write $x^1 \preccurlyeq x^2$ if and only if there exists $y \in \{1, \ldots, M\}$ such that

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c^u_{ijk} x^1_{ijk} = \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c^u_{ijk} x^2_{ijk}, \quad u \in \{1, \ldots, y\}, \tag{7}$$

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c^u_{ijk} x^1_{ijk} < \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c^u_{ijk} x^2_{ijk}, \quad u \in \{1, \ldots, M\} \cap \{y+1\} \tag{8}$$

(i.e., conditions (7) and (8) hold). As a result, the multicriteria problem with the lexicographic convolution of criteria is to find x^* satisfying the system of constraints

$$x^* \in P, \tag{9}$$

$$x^* \preccurlyeq x, \quad x \in P. \tag{10}$$

For the sake of convenience, let $Z_{\preccurlyeq}$ denote problem (9), (10). Obviously, the NP-hard three-index axial assignment problem is polynomially reducible to the problem $Z_{\preccurlyeq}$. In the corresponding problem $Z_{\preccurlyeq}$, we define $c^1_{ijk} = c_{ijk}$ and $c^u_{ijk} = 0$, where $i \in I$, $j \in J$, $k \in K$, and $u = \overline{2, M}$. Then the following result is true.

Proposition 2. *The problem $Z_{\preccurlyeq}$ is NP-hard.*

The minimax convolution of criteria has the form

$$\max_{u\in\{1,\ldots,M\}} \left(\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c^u_{ijk} x_{ijk} \right) \to \min. \tag{11}$$

For the sake of convenience, let Z_{minmax} denote problem (1)–(4), (11). Obviously, the NP-hard three-index axial assignment problem is polynomially reducible to the problem Z_{minmax}. In the corresponding problem Z_{minmax}, we define $c^u_{ijk} = c_{ijk}$, where $i \in I$, $j \in J$, $k \in K$, and $u = \overline{1, M}$. Then the following result is true.

Proposition 3. *The problem Z_{minmax} is NP-hard.*

Thus, the multicriteria axial assignment problem with the linear, lexicographical, or minimax convolution of criteria as a compromise scheme is NP-hard. This raises the question of combining feasible solutions as a complement to heuristic or approximate algorithms for solving these NP-hard problems.

3. THE PROBLEM OF COMBINING SOLUTIONS

Let a given set $W \subseteq I \times J \times K$ define a subset of allowed assignments. Consider problem (1)–(4), (12), (5) with

$$x_{ijk} = 0, \quad (i, j, k) \notin W. \tag{12}$$

For the sake of convenience, let $Z_M(W)$ denote the multicriteria problem (1)–(4), (12), (5) with a given set W. Obviously, problem (1)–(5) corresponds to the problem $Z_M(I \times J \times K)$.

Next, we consider the multicriteria problem $Z_M(W)$ with different criteria convolutions as a compromise scheme.

In the linear convolution case, the corresponding problem takes the form (1)–(4), (12), (6), further denoted by $Z_L(W)$.

In the lexicographic convolution case, the corresponding problem takes the form (9), (10), where the set P is the set of feasible solutions for the system of constraints (1)–(4), (12). It will be denoted by $Z_{\preccurlyeq}(W)$.

In the minimax convolution case, the corresponding problem takes the form (1)–(4), (12), (11), further denoted by $Z_{\text{minmax}}(W)$.

Checking the consistency of the system of constraints in the problem $Z_M(W)$ with an arbitrary set W, i.e., system (1)–(4), (12), is an NP-complete problem [1]. We will consider the sets W corresponding to the assignments of a given subset of feasible solutions.

Let x_{ijk}, where $i \in I$, $j \in J$, and $k \in K$, be a feasible solution of the system of constraints (1)–(4). Then $W(x)$ will denote the following set of allowed assignments:

$$W(x) = \{(i,j,k) | x_{ijk} = 1, i \in I, j \in J, k \in K\}.$$

Let $x^1_{ijk}, \ldots, x^r_{ijk}$, where $i \in I$, $j \in J$, and $k \in K$, be arbitrary r feasible solutions of the system of constraints (1)–(4). Then

$$W(x^1, \ldots, x^r) = W(x^1) \cup \ldots \cup W(x^r).$$

Below, we will investigate the problems

$$Z_L(W(x^1, \ldots, x^r)), \quad Z_{\preccurlyeq}(W(x^1, \ldots, x^r)), \quad Z_{\text{minmax}}(W(x^1, \ldots, x^r)).$$

They are combination problems, i.e., optimization ones on the set of allowed assignments $W(x^1, \ldots, x^r)$, which is built by combining the factual assignments of the selected feasible solutions $x^1, \ldots, x^r$. Thus, the solution of combination problems corresponds to the solution containing only the assignments of the selected feasible solutions.

4. THE LINEAR CONVOLUTION OF CRITERIA

Consider the multicriteria assignment problem in the case of the linear convolution of criteria. Obviously, the problem $Z_L(W(x^1, \ldots, x^r))$ is equivalent to the single-criterion three-index axial problem with the cost matrix

$$c_{ijk} = \sum_{u \in \{1,\ldots,M\}} \alpha_u c^u_{ijk}, \quad i \in I, \quad j \in J, \quad k \in K.$$

Thus, according to [20], the problem $Z_L(W(x^1, x^2))$ is polynomially solvable. The combination algorithm proposed in [20] requires $O(n)$ computational operations and can be applied to solve the problem $Z_L(W(x^1, x^2))$. Based on [22], the following result is true.

Proposition 4. *For $r \geqslant 4$, the class of problems $Z_L(W(x^1, \ldots, x^r))$ is NP-hard.*

For the time being, the complexity status of the class of problems $Z_L(W(x^1, x^2, x^3))$ remains unknown.

5. THE LEXICOGRAPHIC CONVOLUTION OF CRITERIA

We present an algorithm for solving the multicriteria assignment problem in the case of the lexicographic convolution of criteria when combining two feasible solutions.

Algorithm 1 (solution of the problem $Z_{\preccurlyeq}(W(x^1, x^2))$).

Step 1. Construct a graph $G = (V, A)$, where

$$V = \{I \cup J \cup K\}, \quad A = \Big\{(i, j), (i, k), (j, k) | (i, j, k) \in W(x^1, x^2)\Big\}.$$

Step 2. Find the connected components V_l, $l = \overline{1, q}$, of the graph G and construct the subgraphs $G_l = (V_l, A_l)$, $l = \overline{1, q}$, generated by the corresponding connected components.

Step 3. Construct the sets

$$D_l^1 = \Big\{(i, j, k) | (i, j, k) \in W(x^1), (i, j), (i, k), (j, k) \in A_l\Big\}, \quad l = \overline{1, q},$$
$$D_l^2 = \Big\{(i, j, k) | (i, j, k) \in W(x^2), (i, j), (i, k), (j, k) \in A_l\Big\}, \quad l = \overline{1, q}.$$

Step 4. Let

$$P_l^1 = \left\{ p | p = \operatorname*{argmin}_{p \in \{1,2\}} \sum_{(i,j,k) \in D_l^p} c_{ijk}^1 \right\},$$

$$P_l^u = \left\{ p | p = \operatorname*{argmin}_{p \in P_l^{u-1}} \sum_{(i,j,k) \in D_l^p} c_{ijk}^u \right\}, \quad u = \overline{2, M-1},$$

$$p_l^* = \operatorname*{argmin}_{p \in P_l^{M-1}} \sum_{(i,j,k) \in D_l^p} c_{ijk}^M, \quad l = \overline{1, q}.$$

Step 5. Determine the solution of the problem $Z_{\preccurlyeq}(W(x^1, x^2))$ using the following algorithm. Let $x_{ijk}^* := 0$, where $i \in I$, $j \in J$, and $k \in K$. For each $l = \overline{1, q}$, execute

$$x_{ijk}^* := 1, \quad (i, j, k) \in D_l^{p_l^*}.$$

The value of the criteria (5) on a solution x^* is given by

$$\sum_{l=1}^{q} \sum_{(i,j,k) \in D_l^{p_l^*}} c_{ijk}^u, \quad u = \overline{1, M}.$$

Theorem 1. *Algorithm 1 outputs the solution of the problem* $Z_{\preccurlyeq}(W(x^1, x^2))$.

Proof. Assume on the contrary that the output of Algorithm 1 is not a solution of the problem $Z_{\preccurlyeq}(W(x^1, x^2))$. Then there exists a feasible solution x of the problem $Z_{\preccurlyeq}(W(x^1, x^2))$ such that the condition $x^* \preccurlyeq x$ is false.

In each connected component V_l, $l = \overline{1, q}$, any feasible solution of the problem $Z_{\preccurlyeq}(W(x^1, x^2))$ may contain assignments consisting of triplets of only the first or only the second solution; see the proof of [20, Theorem 1].

By the assumption above, there exists a connected component V_l in which the assignments of the solution x do not coincide with the assignments of the solution x^*. Then we construct the solution x' as follows:

Step 1. $x^0 = x,\ l = 1.$

Step 2. $x^t_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \in D_l^{p_l^*} \\ 0 \text{ if } (i,j,k) \in D_l^{3-p_l^*} \\ x^{t-1}_{ijk} \text{ otherwise,} \end{cases} \quad i \in I,\ j \in J,\ k \in K,$

Step 3. If $l = q$, then stop; otherwise, $l = l + 1$ and return to Step 2.

By the construction procedure, $x^{l+1} \preccurlyeq x^l$, $l = \overline{0, q-1}$. In addition, $x^q = x^*$ and $x^0 = x$. Hence, we arrive at the contradiction $x^* \preccurlyeq x$, and the proof of Theorem 1 is complete.

According to [20, Theorem 2], the following result is true.

Proposition 5. *Algorithm 1 requires $O(n)$ computational operations.*

Obviously, the class of three-index axial problems with the set of allowed assignments $W(x^1, x^2, x^3, x^4)$ is polynomially reducible to the class of problems $Z_{\preccurlyeq}(W(x^1, x^2, x^3, x^4))$. As was established in [22], the former class is NP-hard. Indeed, consider the three-index axial assignment problem with the cost matrix c_{ijk}, where $i \in I$, $j \in J$, and $k \in K$. When performing reduction in the corresponding problem $Z_{\preccurlyeq}(W(x^1, x^2, x^3, x^4))$, we define $c^1_{ijk} = c_{ijk}$ and $c^u_{ijk} = 0$, where $i \in I$, $j \in J$, $k \in K$, and $u = \overline{2, M}$. Then the following result is true.

Proposition 6. *For $r \geqslant 4$, the class of problems $Z_{\preccurlyeq}(W(x^1, \ldots, x^r))$ is NP-hard.*

For the time being, the complexity status of the class of problems $Z_{\preccurlyeq}(W(x^1, x^2, x^3))$ remains unknown.

6. THE MINIMAX CONVOLUTION OF CRITERIA

Consider the multicriteria assignment problem in the case of the minimax convolution of criteria.

Lemma 1. *The optimization problem*

$$x'_i \in \{0, 1\}, \quad i = \overline{1, n}, \tag{13}$$

$$\max\left(\sum_{i=1}^{n} a_i x'_i, \sum_{i=1}^{n} b_i (1 - x'_i)\right) \to \min \tag{14}$$

is NP-hard.

Proof. To prove this lemma, we polynomially reduce the classical NP-complete PARTITION problem [5] to problem (13), (14). Consider the PARTITION problem with the initial parameters w_i, $i = \overline{1, m}$, letting $n = m$ and $a_i = b_i = w_i$, $i = \overline{1, n}$. The optimal value of the criterion of problem (13), (14) is $\frac{1}{2} \sum_{i=1}^{m} w_i$ if and only if the PARTITION problem has a solution. Therefore, problem (13), (14) is NP-hard. The proof of Lemma 1 is complete.

Theorem 2. *The class of problems $Z_{\text{minmax}}(W(x^1, x^2))$ is NP-hard.*

Proof. We show the polynomial reducibility of the NP-hard problem (13), (14) to the class of problems $Z_{\text{minmax}}(W(x^1, x^2))$.

Let $N = 2n$, $I = J = K = \{1, \ldots, N\}$, and $M = 2$. We define the three-index cost matrices c^1_{ijk} and c^2_{ijk}, where $i \in I$, $j \in J$, and $k \in K$, as follows:

$$c^1_{ijk} = \begin{cases} a_q \text{ if } \exists q \in \{1, \ldots, n\} \text{ such that } i = j = k = 2q - 1 \\ 0 \quad \text{otherwise,} \end{cases} \quad i \in I,\ j \in J,\ k \in K,$$

$$c^2_{ijk} = \begin{cases} b_q \text{ if } \exists q \in \{1, \ldots, n\} \text{ such that } i = j = k + 1 = 2q \\ 0 \quad \text{otherwise,} \end{cases} \quad i \in I,\ j \in J,\ k \in K.$$

We construct two subsets $P_1, P_2 \subseteq I \times J \times K$ determining two feasible solutions of system (1)–(4):

$$P_1 = \{(i,i,i) | i = \overline{1,N}\},$$
$$P_2 = \{(2i-1, 2i-1, 2i), (2i, 2i, 2i-1) | i = \overline{1,n}\}.$$

We define the corresponding two feasible solutions x^1, x^2 of system (1)–(4) by

$$x_{ijk}^t = \begin{cases} 1 & \text{if } (i,j,k) \in P_t \\ 0 & \text{otherwise,} \end{cases} \quad i \in I,\ j \in J,\ k \in K,\ t \in \{1,2\}.$$

Next, consider the corresponding combination problem $Z_{\text{minmax}}(W(x^1, x^2))$. It is necessary to demonstrate that the optimal value of the criterion of problem (13), (14) coincides with that of problem $Z_{\text{minmax}}(W(x^1, x^2))$.

1. Let x'^* be the optimal solution of problem (13), (14). We construct $P(x'^*)$ using the following algorithm:

Step 1. $P(x'^*) = \emptyset$.

Step 2. For each $i = \overline{1,n}$:
if $x_i'^* = 1$, then $P(x'^*) = P(x'^*) \cup \{(2i-1, 2i-1, 2i-1), (2i, 2i, 2i)\}$;
otherwise, $P(x'^*) = P(x'^*) \cup \{(2i-1, 2i-1, 2i), (2i, 2i, 2i-1)\}$.

As is easily verified, the value of the criterion of the problem $Z_{\text{minmax}}(W(x^1, x^2))$ on the solution corresponding to $P(x'^*)$ coincides with that of problem (13), (14). Assume on the contrary that the solution corresponding to $P(x'^*)$ is not optimal in the problem $Z_{\text{minmax}}(W(x^1, x^2))$. Let x^* be the optimal solution of the problem $Z_{\text{minmax}}(W(x^1, x^2))$. Then the value of the criterion of the problem $Z_{\text{minmax}}(W(x^1, x^2))$ on the solution x^* is strictly smaller than that of problem (13), (14) on the solution x'^*. Then we construct a feasible solution x' of problem (13), (14) as follows:

$$x_i' = \begin{cases} 1 & \text{if } x_{iii}^* = 1 \\ 0 & \text{otherwise,} \end{cases} \quad i = \overline{1,n}.$$

By the construction procedure, the value of the criterion of problem (13), (14) on the solution x' coincides with that of the problem $Z_{\text{minmax}}(W(x^1, x^2))$ on the solution x^*. Consequently, x'^* is not an optimal solution of problem (13), (14). This contradiction proves that the value of the criterion of the problem $Z_{\text{minmax}}(W(x^1, x^2))$ on the solution corresponding to $P(x'^*)$ will coincide with that of the problem (13), (14).

2. Let x^* be the optimal solution to the problem $Z_{\text{minmax}}(W(x^1, x^2))$. Then we construct the optimal solution of problems (13), (14) as follows:

$$x_i' = \begin{cases} 1 & \text{if } x_{iii}^* = 1 \\ 0 & \text{otherwise,} \end{cases} \quad i = \overline{1,n}.$$

As is easily verified, the value of the criterion of problem (13), (14) on the solution x' coincides with the optimal criterion of the problem $Z_{\text{minmax}}(W(x^1, x^2))$.

Assume on the contrary that the solution x' is not optimal in problem (13), (14). Let x'^* be the optimal solution of problem (13), (14). Then the value of the criterion of problem (13), (14) on the solution x'^* is strictly smaller than that of the problem $Z_{\text{minmax}}(W(x^1, x^2))$ on the solution x^*. We construct the solution of the problem $Z_{\text{minmax}}(W(x^1, x^2))$ using the following algorithm:

Step 1. $P(x'^*) = \emptyset$.

Step 2. For each $i = \overline{1,n}$:
if $x_i'^* = 1$, then $P(x'^*) = P(x'^*) \cup \{(2i-1, 2i-1, 2i-1), (2i, 2i, 2i)\}$;
otherwise, $P(x'^*) = P(x'^*) \cup \{(2i-1, 2i-1, 2i), (2i, 2i, 2i-1)\}$.

By the construction procedure, the value of the criterion of the problem $Z_{\mathrm{minmax}}(W(x^1, x^2))$ on the solution corresponding to $P(x'^*)$ coincide with that of problem (13), (14) on the solution x'^*. Therefore, x^* is not the optimal solution of the problem $Z_{\mathrm{minmax}}(W(x^1, x^2))$. This contradiction proves that the value of the criterion of problem (13), (14) on the solution x' will coincide with the optimal value of the criterion of the problem $Z_{\mathrm{minmax}}(W(x^1, x^2))$.

Thus, problem (13), (14) is polynomially reducible to the class of problems $Z_{\mathrm{minmax}}(W(x^1, x^2))$. Consequently, the class of problems $Z_{\mathrm{minmax}}(W(x^1, x^2))$ is NP-hard. The proof of Theorem 2 is complete.

7. A COMPUTATIONAL EXPERIMENT

Consider the problem $Z_{\preccurlyeq}$ for $M = 2$. By analogy with [12], we construct a test set with three-index cost matrices whose elements are random integer values with the uniform distribution on the interval [0, 300]. Let us conduct a series of experiments for fixed values of n, denoting by K the number of problems in the series. We design two heuristic algorithms for solving the problem $Z_{\preccurlyeq}$. The first algorithm is based on constructing subsets of feasible solutions and selecting a record among them; the second one is similar to the first, but the record selection step is replaced by the sequential combination of solutions using the first algorithm. Note that by Proposition 5, the first algorithm has a complexity of $O(n)$, which coincides with the complexity of record selection.

The first heuristic algorithm. Construct $N = n^3$ random solutions, and apply the local optimization algorithm [13] to each of them. Denote by x'_t, $t = \overline{1, N}$, the resulting feasible solutions of the problem $Z_{\preccurlyeq}$. Among them, choose the record $x^* : x^* \preccurlyeq x'_t$, $t = \overline{1, N}$, as the output of the algorithm.

The second heuristic algorithm. Replace the record selection step of the first heuristic algorithm with the sequential combination of the pairs of solutions as follows. Let x''_1 be the solution of the problem $Z_{\preccurlyeq}(W(x'_1, x'_2))$. Denote by x''_t the solution of the problem $Z_{\preccurlyeq}(W(x''_{t-1}, x'_{t+1}))$, $t = \overline{2, N-1}$. Choose x''_{N-1} as the output of the algorithm.

Denoting by $C^1(x)$ and $C^2(x)$ the values of the criteria (5) on the solution x, we will compare the average deviation of the criteria values under combination (the second heuristic algorithm) and record choice (the first heuristic algorithm) in the series. The results are presented in Table 1.

Table 1

n	K	$100\% \dfrac{C^1(x^*) - C^1(x''_{N-1})}{C^1(x^*)}$	$100\% \dfrac{C^2(x^*) - C^2(x''_{N-1})}{C^2(x^*)}$
10	10	3.09%	1.98%
11	10	5.3%	−6.23%
12	10	5.31%	6.58%
13	10	2.05%	0.46%
14	10	0%	0%
15	10	1.26%	−1.85%
16	10	2.5%	2.14%
17	10	2.74%	−4.92%
18	10	3.19%	−0.7%
19	10	7.08%	2.3%

Thus, the average deviations over all series are 3.94% and 0.7% for the first and second criteria, respectively. This demonstrates the effectiveness of the combination strategy instead of the generally accepted record selection strategy.

The linear convolution case is equivalent to the single-criterion problem with the linear criterion and was considered in [20, 21], including the results of a computational experiment. The minimax convolution case is not considered in the computational experiment here: by Theorem 2, the combination problem with the minimax convolution is NP-hard even when combining two solutions.

8. CONCLUSIONS

Solution algorithms for the NP-hard multicriteria three-index axial assignment problem with different types of criteria convolutions have been studied. The problem of combining feasible solutions of this problem as a compromise scheme has been formulated. Solutions can be combined as a complement to known heuristics or approximate algorithms for post-processing the obtained approximate solutions of the assignment problem instead of the generally accepted practice of record selection.

The complexity status of combining solutions has been investigated. It has been shown that, in the case of the linear or lexicographic convolution, the pairs of solutions can be completely combined in a time of $O(n)$; the class of combination problems for four or more solutions is NP-hard; the complexity status of combining three solutions is an open issue. In the minimax convolution case, the class of combination problems for two or more solutions is NP-hard. For better clarity, the outcomes of this paper are presented in Table 2.

Table 2

	$Z_L(W(x^1,\ldots,x^r))$	$Z_{\preccurlyeq}(W(x^1,\ldots,x^r))$	$Z_{\text{minmax}}(W(x^1,\ldots,x^r))$
$r=2$	$O(n)$	$O(n)$	NP-hard
$r=3$	?	?	
$r\geqslant 4$	NP-hard	NP-hard	

In addition, an algorithm for combining solution pairs has been designed in the lexicographic convolution case. According to the computational experiment, the combination strategy allows decreasing the deviation from the optimum as compared to the record selection strategy.

Further research will address the open cases of combining three solutions and combining solutions when constructing Pareto-optimal solutions of the multicriteria assignment problem.

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